MODELING OF STEADY FLOWS IN A CHANNEL BY NAVIER–STOKES VARIATIONAL INEQUALITIES

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A mathematical model of a steady viscous incompressible fluid flow in a channel with exit conditions different from the Dirichlet conditions is considered. A variational inequality is derived for the formulated subdifferential boundary-value problem, and the structure of the set of its solutions is studied. For two-dimensional problems, the solvability of the problem without the assumption on the low Reynolds number is proved. In the three-dimensional case, a class of constraints on the tangential component of velocity at the exit, which guarantees solvability of the variational inequality, is found.

Key words: *Navier–Stokes equations, boundary conditions, steady flows, variational inequalities.*

Introduction. We consider the motion of a uniform incompressible viscous fluid in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, which is described by Navier–Stokes equations for the vector function of flow velocity $\boldsymbol{u}(x)$ and the scalar function of pressure $p(x)$:

$$\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \boldsymbol{f}, \qquad \operatorname{div}\boldsymbol{u} = 0, \qquad x \in \Omega. \tag{1}$$

Here, $\nu > 0$ is the kinematic viscosity and $\boldsymbol{f}$ is the vector of density of external body forces; the fluid density is assumed to equal unity.

Let Γ_1 be part of the boundary Γ of the domain Ω through which the fluid flows in, Γ_0 be the solid wall, and Γ_2 be part of the boundary through which the fluid can leave the domain Ω. A typical example of Ω is a channel with an inflow sector Γ_1 and outflow sector Γ_2.

One can naturally set the velocity vector at the sectors Γ_0 and Γ_1:

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma_1, \qquad \boldsymbol{u} = 0 \quad \text{on } \Gamma_0. \tag{2}$$

Here $\boldsymbol{g} = \boldsymbol{g}(x)$ $(x \in \Gamma_1)$ is a given function and $g_n = \boldsymbol{g}\cdot\boldsymbol{n} < 0$ ($\boldsymbol{n}$ is a unit vector of the external normal to the boundary). The fluid behavior on the sector Γ_2 depends on the flow structure, which is *a priori* unknown; therefore, the question of imposing boundary conditions on Γ_2 arises.

In the present work, we set the following boundary conditions on the sector Γ_2:

$$\boldsymbol{n}\times\operatorname{rot}\boldsymbol{u} \in \partial\Psi(\boldsymbol{u}_\tau), \qquad \boldsymbol{u}\cdot\boldsymbol{n} = q \qquad \text{on } \Gamma_2. \tag{3}$$

Here $\boldsymbol{u}_\tau = \boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}$, Ψ: $\mathbb{R}^d \to \overline{\mathbb{R}} = (-\infty, +\infty]$ is a given function possessing the properties of convexity and weak semi-continuity from below $(\Psi \not\equiv +\infty)$. The set $\partial\Psi(\boldsymbol{a})$ is a subdifferential of the function Ψ at the point $\boldsymbol{a}$:

$$\partial\Psi(\boldsymbol{a}) = \{\boldsymbol{b} \in \mathbb{R}^d\colon \quad \Psi(\boldsymbol{h}) - \Psi(\boldsymbol{a}) \geqslant \boldsymbol{b}\cdot(\boldsymbol{h} - \boldsymbol{a}) \quad \forall \boldsymbol{h} \in \mathbb{R}^d\}.$$

The study of formulation of problem (1)–(3) is based on investigating the Navier–Stokes-type variational inequality obtained in Sec. 1. Based on the results of [1, 2], we derive conditions of solvability of the inequality and describe the structure of the set of solutions in the case of nonuniqueness. As particular cases, problem (1)–(3) contains problem formulations with the values of $\boldsymbol{u}_\tau$ or $\boldsymbol{n}\times\operatorname{rot}\boldsymbol{u}$ specified on Γ_2. Solvability of such problems with nonzero values of tangential components of the velocity vector was proved previously only for low Reynolds numbers [3]. Note also, various boundary conditions at the channel exit, leading to variational inequalities, were considered in [4–7].

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The subdifferential boundary condition on the outflow sector in the form (3) is caused by the formulation of the flow problem where the tangential components of the velocity vector are not set explicitly. Instead, a variational principle is formulated, which determined the "missing" boundary condition. In this case, the variational principle has the form

$$E(\boldsymbol{v}) = \{\Psi(\boldsymbol{v}) + (\operatorname{rot}\boldsymbol{u}\times\boldsymbol{n})\boldsymbol{v}\} \mapsto \inf \quad \forall\boldsymbol{v}, \quad \boldsymbol{v}\cdot\boldsymbol{n} = 0; \tag{4}$$

$$E(\boldsymbol{u}_\tau) = \inf E(\boldsymbol{v}).$$

Relation (4) with different choices of the function Ψ allows one to model different dependences between flow vorticity at the boundary $\operatorname{rot}\boldsymbol{u}\times\boldsymbol{n}$ and tangential components of velocity, which take into account energy transfer on the sector Γ_2.

Let us consider several examples.

Example 1. Let

$$\Psi(\boldsymbol{v}) = \boldsymbol{l}\cdot\boldsymbol{v},$$

where $\boldsymbol{l} = \boldsymbol{l}(x)$ $(x\in\Gamma_2)$ is a vector function specified on Γ_2, and $\boldsymbol{l}\cdot\boldsymbol{n} = 0$. In this case, we obtain the boundary condition

$$\boldsymbol{n}\times\operatorname{rot}\boldsymbol{u} = \boldsymbol{l}, \qquad x\in\Gamma_2. \tag{5}$$

Example 2. Let

$$\Psi(\boldsymbol{v}) = \lambda\boldsymbol{v}^2/2,$$

where $\lambda = \lambda(x) \geqslant 0$ $(x\in\Gamma_2)$ is a scalar function specified on Γ_2. Then, we obtain the boundary condition

$$\boldsymbol{n}\times\operatorname{rot}\boldsymbol{u} = \lambda\boldsymbol{u}_\tau, \qquad x\in\Gamma_2. \tag{6}$$

Example 3. Let

$$\Psi(\boldsymbol{v}) = \begin{cases} 0, & \boldsymbol{v} = \boldsymbol{l}, \\ +\infty, & \boldsymbol{v} \neq \boldsymbol{l}, \end{cases}$$

where $\boldsymbol{l} = \boldsymbol{l}(x)$ $(x\in\Gamma_2)$ is a vector function specified on Γ_2, and $\boldsymbol{l}\cdot\boldsymbol{n} = 0$. Then, condition (3) is equivalent to setting the full velocity vector on Γ_2:

$$\boldsymbol{u}_\tau = \boldsymbol{l}, \qquad \boldsymbol{u}_n = q, \qquad x\in\Gamma_2. \tag{7}$$

Example 4. Let

$$\Psi(\boldsymbol{v}) = \begin{cases} \lambda\boldsymbol{v}^2/2, & |\boldsymbol{v}| \leqslant v_0, \\ +\infty, & |\boldsymbol{v}| > v_0, \end{cases}$$

where $v_0 > 0$ is a given number limiting the absolute value of the tangential component of velocity. Then, the subdifferential condition (3) yields the following constraints on Γ_2:

$$\boldsymbol{n}\times\operatorname{rot}\boldsymbol{u} = \begin{cases} \lambda\boldsymbol{u}_\tau, & |\boldsymbol{u}_\tau| < v_0, \\ (\lambda+\sigma)\boldsymbol{u}_\tau, & |\boldsymbol{u}_\tau| = v_0. \end{cases} \tag{8}$$

Here $\sigma = \sigma(x) \geqslant 0$ is a scalar function (unknown *a priori*).

1. Formalization of the Boundary-Value Problem. Let the flow domain (channel) be bounded and simply connected in $\mathbb{R}^d$ $(d = 2,\ 3)$. We assume that Γ_0, Γ_1, and Γ_2 are nonzero open subsets of the connected boundary $\Gamma = \partial\Omega = \overline{\Gamma_0}\cup\overline{\Gamma_1}\cup\overline{\Gamma_2}$ $(\Gamma_i\cap\Gamma_j = \varnothing,\ i\neq j)$. The sector Γ_2 is a union of a finite number of simple smooth surfaces (curves if $d = 2$) bordering on Γ_0, and the set Γ–Γ_0–Γ_1–Γ_2 consists of a finite number of simple closed smooth curves for $d = 3$.

We consider the following space of vector functions defined on Ω:

$$\boldsymbol{V} = \{\boldsymbol{v}\in\boldsymbol{W}_2^1(\Omega):\quad \operatorname{div}\boldsymbol{v} = 0 \quad \text{in } \Omega; \quad \boldsymbol{v} = 0 \quad \text{on } \Gamma_0\}.$$

Here $\boldsymbol{W}_r^m$ are the Sobolev spaces and $\boldsymbol{W}_2^m = \boldsymbol{H}^m$. The scalar product in the Hilbert space $\boldsymbol{V}$ is determined by the formula

$$(\boldsymbol{u},\boldsymbol{v}) = \sum_1^d \int\limits_\Omega \nabla u_i\cdot\nabla v_i\,dx, \qquad \|\boldsymbol{u}\|^2 = (\boldsymbol{u},\boldsymbol{u}).$$

In the formulation of the boundary-value problem (1)–(3), the vector functions $\boldsymbol{f} = \boldsymbol{f}(x)$ $(x \in \Omega)$ and $\boldsymbol{g} = \boldsymbol{g}(x)$ $(x \in \Gamma_1)$ and the scalar function $q = q(x)$ $(x \in \Gamma_2)$ are given, and

$$\int\limits_{\Gamma} \tilde{q}\, dS = 0, \qquad \tilde{q} = \begin{cases} \boldsymbol{g}\cdot\boldsymbol{n}, & x \in \Gamma_1, \\ 0, & x \in \Gamma_0, \\ q, & x \in \Gamma_2. \end{cases}$$

If

$$\boldsymbol{g}\cdot\boldsymbol{n} \leqslant 0, \quad x \in \Gamma_1, \qquad q \geqslant 0, \quad x \in \Gamma_2,$$

then, Γ_1 is the sector of fluid inflow, Γ_2 is the sector of fluid outflow, and Γ_0 is the solid wall of the channel. We assume that the function $\boldsymbol{g}$ is the trace of a certain function from the space $\boldsymbol{V}$ on Γ_1 and the function q is the trace of the normal component on Γ_2; the function

$$\tilde{\boldsymbol{g}} = \begin{cases} \boldsymbol{g}(x), & x \in \Gamma_1, \\ 0, & x \in \Gamma_0 \cup \Gamma_2 \end{cases}$$

belongs to the class $H^{1/2}(\Gamma)$. We denote the space conjugate to $\boldsymbol{V}$ as $\boldsymbol{V}'$ and define the following operators:

$$A\colon \boldsymbol{V} \to \boldsymbol{V}', \qquad (A\boldsymbol{u}, \boldsymbol{v}) = \nu(\operatorname{rot}\boldsymbol{u}, \operatorname{rot}\boldsymbol{v})_{L^2(\Omega)} + (u_n, v_n)_{H^{1/2}(\Gamma)},$$

$$B\colon \boldsymbol{V}\times\boldsymbol{V} \to \boldsymbol{V}', \qquad B[\boldsymbol{u}] = B(\boldsymbol{u},\boldsymbol{u}), \qquad (B(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}) = \int\limits_{\Omega} (\operatorname{rot}\boldsymbol{u}\times\boldsymbol{v})\boldsymbol{w}\, dx,$$

$$\Phi(\boldsymbol{v}) = \begin{cases} \nu\displaystyle\int\limits_{\Gamma_2} \Psi(\boldsymbol{v}_\tau)\, ds, & \Psi(\boldsymbol{v}_\tau(x)) \in L^1(\Gamma_2), \\ +\infty, & \Psi(\boldsymbol{v}_\tau(x)) \notin L^1(\Gamma_2). \end{cases}$$

We denote the value of the functional $\boldsymbol{h} \in \boldsymbol{V}'$ on the element $\boldsymbol{w} \in \boldsymbol{V}$ as $(\boldsymbol{h}, \boldsymbol{w})$. Let

$$\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{V}, \quad \boldsymbol{v} = \boldsymbol{g} \quad \text{on } \Gamma_1, \quad \boldsymbol{v} = 0 \quad \text{on } \Gamma_0, \quad \boldsymbol{v}\cdot\boldsymbol{n} = q \quad \text{on } \Gamma_2\} \cap \operatorname{dom}\Phi,$$

where $\operatorname{dom}\Phi$ consists of elements of the space $\boldsymbol{V}$ on which the functional Φ is finite.

Definition 1. The vector function $\boldsymbol{u} \in \boldsymbol{K}$ is called the generic solution of problem (1)–(3) if

$$(A\boldsymbol{u} + B[\boldsymbol{u}] - \boldsymbol{f}, \boldsymbol{u} - \boldsymbol{v}) + \Phi(\boldsymbol{u}) - \Phi(\boldsymbol{v}) \leqslant 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K}. \tag{9}$$

Indeed, if $\{\boldsymbol{u}, p\}$ is a rather smooth solution of the boundary-value problem (1)–(3), then, multiplying the equation of momenta written in the Lamb form by $\boldsymbol{u} - \boldsymbol{v}$ and integrating over the domain Ω, we obtain

$$\nu(\operatorname{rot}\boldsymbol{u}, \operatorname{rot}(\boldsymbol{u}-\boldsymbol{v})) + \nu\int\limits_{\Gamma_2} (n\times\operatorname{rot}\boldsymbol{u})(\boldsymbol{u}_\tau - \boldsymbol{v}_\tau)\, ds + (B[\boldsymbol{u}], \boldsymbol{u}-\boldsymbol{v}) = (f, \boldsymbol{u}-\boldsymbol{v}).$$

Using the inequality following from (3),

$$(\boldsymbol{n}\times\operatorname{rot}\boldsymbol{u})(\boldsymbol{v}_\tau - \boldsymbol{u}_\tau) \leqslant \Psi(\boldsymbol{v}_\tau) - \Psi(\boldsymbol{u}_\tau),$$

we obtain (9). The reverse is also valid: if $\boldsymbol{u} \in \boldsymbol{K}$ is a rather smooth vector function, then, assuming in (9) that

$$\boldsymbol{v} = \boldsymbol{u} \pm \boldsymbol{w} \quad \forall \boldsymbol{w} \in \overset{\circ}{C}{}^{\infty}(\Omega), \qquad \operatorname{div}\boldsymbol{w} = 0,$$

we obtain

$$\nu(\operatorname{rot}\boldsymbol{u}, \operatorname{rot}\boldsymbol{w})_{L^2(\Omega)} + (B[\boldsymbol{u}], \boldsymbol{w}) = (\boldsymbol{f}, \boldsymbol{w}).$$

It follows from here that the vector $\boldsymbol{u}$ satisfies the equation of momenta in Ω, from which, by multiplying by $\boldsymbol{u} - \boldsymbol{v}$, integrating over Ω, and comparing with (9), we obtain the inequality

$$\nu\int\limits_{\Gamma_2} \Big(\Psi(\boldsymbol{v}_\tau) - \Psi(\boldsymbol{u}_\tau) - (\boldsymbol{n}\times\operatorname{rot}\boldsymbol{u})(\boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\Big)\, dS \geqslant 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K},$$

which confirms the validity of the subdifferential boundary condition (3).

2. Solvability of the Variational Inequality. Let us study the properties of the operators A and B and the functional Φ. Note that

$$(A\boldsymbol{u},\boldsymbol{v}) = (A\boldsymbol{v},\boldsymbol{u}), \qquad (B(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{v}) = 0 \qquad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}.$$

One can easily verify (see, e.g., [1, 2]) that

$$(A\boldsymbol{u},\boldsymbol{u}) \geqslant \gamma \|\boldsymbol{u}\|^2 \quad \forall \boldsymbol{u} \in \boldsymbol{V},$$

where $\gamma > 0$ is independent of $\boldsymbol{u}$; the image of $B[\boldsymbol{u}]$: $\boldsymbol{V} \to \boldsymbol{V}'$ is strongly continuous, i.e., converts sequences weakly converging in $\boldsymbol{V}$ to sequences strongly converging in $\boldsymbol{V}'$. The functional Φ is convex and semicontinuous from below. Therefore, to prove the solvability of the variational inequality (9), it suffices to learn under which conditions the nonlinear operator $B[\boldsymbol{u}]$ is "subordinated" to the operator A on the set $\boldsymbol{K}$, i.e., obtain the following estimate: for all $\delta > 0$, it is possible to find an element $\boldsymbol{w}_\delta \in \boldsymbol{K}$ such that

$$(B[\boldsymbol{u}],\boldsymbol{w}_\delta) \leqslant \delta \|\boldsymbol{u}\|^2 + C_\delta \quad \forall \boldsymbol{u} \in \boldsymbol{K}. \tag{10}$$

Here $C_\delta > 0$ is independent of $\boldsymbol{u} \in \boldsymbol{K}$.

Condition (10) establishes a relation between the "structure" of the set $\boldsymbol{K}$ and the quadratic operator $B[\boldsymbol{u}]$. Verification of this condition is one of the main stages of investigation of particular problems in hydrodynamics. This condition, nevertheless, is obviously satisfied if the set $\boldsymbol{K}$ contains a zero element or is bounded.

2.1. *Two-Dimensional Flows.* In the two-dimensional case, the nonlinear operator $B[\boldsymbol{u}]$: $\boldsymbol{V} \to \boldsymbol{V}'$ is determined by the expression

$$(B[\boldsymbol{u}],\boldsymbol{w}) = \int\limits_\Omega \omega Z(\boldsymbol{u}) \cdot \boldsymbol{w}\, dx,$$

where $\omega = \operatorname{rot} \boldsymbol{u} \equiv \partial u_2/\partial x_1 - \partial u_1/\partial x_2$ is flow vorticity and $Z(\boldsymbol{u}) = \{-u_2, u_1\}$ is the rotation of the vector $\boldsymbol{u} = \{u_1, u_2\}$ by $\pi/2$.

Note that any vector function $\boldsymbol{u} \in \boldsymbol{K}$ can be presented in the form

$$\boldsymbol{u} = \boldsymbol{v} + \nabla r,$$

where $r \in W_2^2(\Omega)$ is a solution of the problem

$$\Delta r = 0, \quad x \in \Omega, \qquad \frac{\partial r}{\partial n} = \tilde{q} \quad \text{on } \Gamma.$$

Then, the function $\boldsymbol{v}$ has a zero normal component on Γ. Thus, to prove (10), it suffices to estimate the expression $(B[\boldsymbol{v}],\boldsymbol{w})$, where $\boldsymbol{w} \in \boldsymbol{K}$. Let $\boldsymbol{w}_0 = \operatorname{Rot} b = \{\partial b/\partial x_2, -\partial b/\partial x_1\}$ be an element of the set $\boldsymbol{K}$, and $b \in W_2^2(\Omega)$ be a scalar function.

Following [8], we determined the "patch" function by the relation

$$\mu_\varepsilon(\lambda) = \begin{cases} 1, & 0 \leqslant \lambda < \sigma(\varepsilon)^2, \\ \varepsilon \ln(\sigma(\varepsilon)/\lambda), & \lambda \in (\sigma(\varepsilon)^2, \sigma(\varepsilon)), \\ 0, & \lambda > \sigma(\varepsilon). \end{cases}$$

Here $\sigma(\varepsilon) = \exp(-1/\varepsilon)$. Smoothing of the continuous function $\mu_\varepsilon(\lambda)$ yields a twice continuously differentiable function $\tilde{\mu}_\varepsilon(\lambda)$ such that

$$\tilde{\mu}_\varepsilon(\lambda) = 1, \quad \lambda \in [0, \sigma(\varepsilon)^2/2], \qquad \tilde{\mu}_\varepsilon(\lambda) = 0, \quad \lambda > 2\sigma(\varepsilon),$$

$$|\tilde{\mu}'_\varepsilon(\lambda)| \leqslant 2\varepsilon/\lambda, \qquad \lambda > 0.$$

We assume that $\theta_\varepsilon(x) = \tilde{\mu}_\varepsilon(d_\Gamma(x))$, where $d_\Gamma(x)$ is the distance from x to the boundary Γ, and

$$\boldsymbol{w}_\varepsilon = \operatorname{Rot}(\theta_\varepsilon b) = b \operatorname{Rot} \theta_\varepsilon + \theta_\varepsilon \operatorname{Rot} b \in \boldsymbol{K}$$

if the number $\varepsilon > 0$ is sufficiently small. Then, we have

$$(B[\boldsymbol{v}],\boldsymbol{w}_\varepsilon) = \int\limits_\Omega \operatorname{rot} \boldsymbol{v} \cdot Z(\boldsymbol{v})(b \operatorname{Rot} \theta_\varepsilon + \theta_\varepsilon \operatorname{Rot} b)\, dx$$

$$\leqslant -\int\limits_{\Omega_\varepsilon} b \operatorname{rot} \boldsymbol{v}(\boldsymbol{v} \cdot \nabla \theta_\varepsilon)\, dx + \alpha(\varepsilon) \|\operatorname{rot} \boldsymbol{v}\|_{L^2(\Omega)} \|\boldsymbol{v}\|_{L^4(\Omega)},$$

where $\alpha(\varepsilon) = \|\operatorname{Rot} b\|_{L^4(\Omega_\varepsilon)} \to 0$ as $\varepsilon \to +0$; $\Omega_\varepsilon = \{x \in \Omega,\ d_\Gamma(x) < 2\sigma(\varepsilon)\}$. Estimating the first term, we note that

$$\boldsymbol{v} \cdot \nabla\theta_\varepsilon = \tilde{\mu}'_\varepsilon(d_\Gamma(x))(\boldsymbol{v}(x) \cdot \nabla d_\Gamma(x)),$$

and $\boldsymbol{v}(x) \cdot \nabla d_\Gamma(x) = 0$ on Γ. Hence,

$$\Big|\int\limits_{\Omega_\varepsilon} b \operatorname{rot} \boldsymbol{v}(\boldsymbol{v} \cdot \nabla\theta_\varepsilon)\, dx\Big| \leqslant \int\limits_{\Omega_\varepsilon} |b \operatorname{rot} \boldsymbol{v}|\, |(\boldsymbol{v}\nabla d_\Gamma)| \frac{2\varepsilon}{d_\Gamma}\, dx \leqslant \varepsilon \|b\|_{L^\infty(\Omega)} \|\operatorname{rot} \boldsymbol{v}\|_{L^2(\Omega)} \|\boldsymbol{v}\nabla d_\Gamma\|_{W_2^1(\Omega)}.$$

The last estimate confirms the validity of condition (10). As a result, we obtain the following statement.

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$, the set $\boldsymbol{K}$ be not empty, $\partial\Phi(\boldsymbol{v}) \neq \varnothing\ \forall \boldsymbol{v} \in \boldsymbol{K}$, and $\boldsymbol{f} \in L^2(\Omega)$. Then, there exists a generic solution of problem* (1)–(3).

Remark 1. It follows from Theorem 1 that, in the two-dimensional case for all Reynolds numbers, boundary-value problems with conditions, e.g., of the form (5)–(8) are solvable. Uniqueness is observed for low Reynolds numbers. In the general case, the set of solutions is homeomorphic to a finite-dimensional compact [1].

2.2. *Three-Dimensional Flows with a Limited Tangential Component of Velocity.* In contrast to the two-dimensional case, the validity of condition (10) could be confirmed under the assumption

$$\|\boldsymbol{w}_\tau\|_{L^2(\Gamma_2)} \leqslant \lambda_0 \quad \forall \boldsymbol{w} \in \boldsymbol{K}, \tag{11}$$

where $\lambda_0 > 0$ is independent of $\boldsymbol{w} \in \boldsymbol{K}$.

Condition (11) is satisfied, in particular, for the function $\Psi(\boldsymbol{v})$ in Example 4. In this case, the subdifferential condition (3) yields the boundary condition (8) on the sector Γ_2.

Lemma 1. *Let $\boldsymbol{K} \neq \varnothing$ and condition* (11) *be satisfied. Then, condition* (10) *is satisfied for the operator B on the set $\boldsymbol{K}$.*

Proof. As in the case $d = 2$, it suffices to show that

$$(B(\boldsymbol{u}, \boldsymbol{u}), \boldsymbol{w}_\gamma) \leqslant \gamma\|\boldsymbol{u}\|^2 + C_\gamma \quad \forall \gamma > 0 \quad \exists \boldsymbol{w}_\gamma \in \boldsymbol{K} \tag{12}$$

for an arbitrary vector function $\boldsymbol{u} \in \boldsymbol{V}$ such that $u_n = 0$ on Γ and $\|\boldsymbol{u}_\tau\|_{L^2(\Gamma)} \leqslant \lambda_0$.

First, we consider the following problem: find a function $\boldsymbol{u}_0 \in \boldsymbol{V}$ such that $\boldsymbol{u}_0 \cdot \boldsymbol{n} = 0$ on Γ and

$$-(\boldsymbol{u}_0, \Delta\boldsymbol{v})_{L^2(\Omega)} + \int\limits_\Gamma \operatorname{rot} \boldsymbol{v}(\boldsymbol{u} \times \boldsymbol{n})\, dS = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V} \cap \boldsymbol{W}_2^2(\Omega), \qquad \boldsymbol{v} = 0 \quad \text{on } \Gamma. \tag{13}$$

One can easily see that the function $\boldsymbol{u}_0 \in \boldsymbol{V}$, which is a solution of the Stokes system

$$-\Delta\boldsymbol{u}_0 + \nabla\xi = 0, \quad \operatorname{div} \boldsymbol{u}_0 = 0 \quad \text{in } \Omega, \qquad \boldsymbol{u}_0 = \boldsymbol{u} \quad \text{on } \Gamma,$$

satisfies Eq. (13).

We obtain an estimate for $\boldsymbol{u}_0$ in $L^3(\Omega)$. We choose the solution of the following system as an element $\boldsymbol{v} \in \boldsymbol{V} \cap \boldsymbol{W}_2^2(\Omega)$:

$$-\Delta\boldsymbol{v} + \nabla\xi = |\boldsymbol{u}_0| \cdot \boldsymbol{u}_0, \quad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{v} = 0 \quad \text{on } \Gamma. \tag{14}$$

Note, the right side of the first equation in this system is $\boldsymbol{f} = |\boldsymbol{u}_0| \cdot \boldsymbol{u}_0 \in L^{3/2}(\Omega)$, and $\|\boldsymbol{f}\|_{L^{3/2}(\Omega)} = \|\boldsymbol{u}_0\|^2_{L^3(\Omega)}$. Thus, Eq. (14) yield the estimate

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^2_{3/2}(\Omega)} \leqslant C\|\boldsymbol{u}_0\|^2_{L^3(\Omega)}. \tag{15}$$

Substituting $\boldsymbol{v}$ into (13), on the basis of Eq. (15), we obtain

$$\|\boldsymbol{u}_0\|^3_{L^3(\Omega)} = \int\limits_\Gamma \operatorname{rot} \boldsymbol{v}(\boldsymbol{u} \times \boldsymbol{n})\, dS \leqslant \lambda_0 \|\operatorname{rot} \boldsymbol{v}\|_{L^2(\Gamma)} \leqslant C\lambda_0 \|\boldsymbol{v}\|_{\boldsymbol{W}^2_{3/2}(\Omega)} \leqslant C\lambda_0 \|\boldsymbol{u}_0\|^2_{L^3(\Omega)}.$$

Hence, we obtain the estimate

$$\|\boldsymbol{u}_0\|_{L^3(\Omega)} \leqslant C\lambda_0,$$

where C is independent of $\boldsymbol{u} \in \boldsymbol{V}$, $u_n = 0$ on Γ, and $\|\boldsymbol{u}_\tau\|_{L^2(\Gamma)} \leqslant \lambda_0$.

We show the validity of Eq. (12):

$$(B(\boldsymbol{u},\boldsymbol{u}),\boldsymbol{w}_\gamma) = \int\limits_\Omega (\operatorname{rot}\boldsymbol{u} \times (\boldsymbol{u}-\boldsymbol{u}_0))\boldsymbol{w}_\gamma\,dx + \int\limits_\Omega (\operatorname{rot}\boldsymbol{u} \times \boldsymbol{u}_0)\boldsymbol{w}_\gamma\,dx$$
$$\leqslant \int\limits_\Omega (\operatorname{rot}\boldsymbol{u} \times (\boldsymbol{u}-\boldsymbol{u}_0))\boldsymbol{w}_\gamma\,dx + C\|\boldsymbol{u}\|\lambda_0\|\boldsymbol{w}_\gamma\|_{L^6(\Omega)}.$$

Then, to obtain Eq. (12), it suffices to estimate the first term in the last inequality.

We take into account that $\boldsymbol{u}-\boldsymbol{u}_0 \in \boldsymbol{V}$ has the property $\boldsymbol{u}-\boldsymbol{u}_0 = 0$ on Γ. Then, representing some fixed element $\boldsymbol{w}_0 \in \boldsymbol{K}$ in the form $\boldsymbol{w}_0 = \operatorname{rot}\boldsymbol{b}$ and determining $\boldsymbol{w}_\gamma = \operatorname{rot}(\theta_\varepsilon \boldsymbol{b}) \in \boldsymbol{K}$, where $\theta_\varepsilon \in C^2(\overline{\Omega})$ is a family of patch functions [8], we obtain the estimate

$$\int\limits_\Omega (\operatorname{rot}\boldsymbol{u} \times (\boldsymbol{u}-\boldsymbol{u}_0))\boldsymbol{w}_\gamma\,dx \leqslant \gamma\|\boldsymbol{u}\|\,\|\boldsymbol{u}-\boldsymbol{u}_0\| + C_\gamma,$$

where C_γ is independent of $\boldsymbol{u}$. Satisfaction of this inequality is sufficient for the validity of estimate (12).

Theorem 2. *Let $\boldsymbol{K} \neq \emptyset$, $\partial\Phi(\boldsymbol{v}) \neq \emptyset$ $\forall \boldsymbol{v} \in \boldsymbol{K}$, $\boldsymbol{f} \in L^2(\Omega)$, and condition* (11) *be satisfied. Then, the set of generic solutions of problem* (1)–(3) *is not empty and is homeomorphic to a finite-dimensional compact.*

Remark 2. It should be noted that the sufficient condition for solvability of the variational inequality (9) is the boundedness of the tangential components of velocity in the norm $L^2(\Gamma)$. Having such *a priori* information on the solution of the Navier–Stokes equations [e.g., in the form of the variational principle of the type (4)], one can prove the boundedness of the solution in the norm of the space $\boldsymbol{V}$ and, correspondingly, the boundedness of the tangential components in $H^{1/2}(\Gamma)$.

A rather strong constraint (11) that guarantees solvability of the three-dimensional problem (1)–(3) can be weakened by requiring the boundedness of the tangential components of vector functions from the set $\boldsymbol{K}$ in a weaker norm of the space $\boldsymbol{W}_3^{-1/3}$.

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